

# Momentum Sum Rules in QCD for a Photon Target

L. L. Frankfurt<sup>a,\*</sup> and E. G. Gurvich<sup>b</sup>

*School of Physics and Astronomy*

*Raymond and Beverly Sackler Faculty of Exact Sciences*

*Tel Aviv University, Ramat Aviv 69978, Israel*

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## Abstract

We deduce momentum sum rules for the parton structure functions of a photon target. Non-perturbative QCD contribution to the momentum sum rules follows from conservation of the energy-momentum tensor and it is calculated through the hadronic part of the photon vacuum polarization operator. The contribution of the unresolved photon unambiguously follows from gauge invariance, renormalizability and asymptotic freedom in QCD. We also compare available parametrizations of parton distribution in a photon with the deduced sum rules.

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The aim of this study is to deduce the momentum sum rule (MSR) for the case of parton distributions (PD) in a photon target. A naive expectation based on the parton model approximation is that the integral over  $x$  with the weight  $x$  from the sum of parton distributions of a (quasi)real photon target should be equal to the probability of a target photon to be in a hadron configuration,  $Z^{-1}(\gamma \rightarrow \text{hadrons})$ . This expectation is invalid in QCD and QED. The nontrivial contribution to the photon wave function due to configurations of quarks with large relative transverse momenta (the unresolved photon), can not be described in terms of the parton model.

Throughout this Letter we denote the four momenta of the probing virtual photon and of the target photon by  $q$  and  $p$  respectively. The mass squared of the probing photon (target photon) is  $q^2 = -Q^2$  ( $p^2 = -P^2$ ). With  $\nu = 2qp$ , the Bjorken variable  $x = Q^2/\nu$ . We define the parton distribution functions of flavour  $f$  as  $q_f^{\text{T(L)}} = \frac{F_2^{f\text{T(L)}}(x, Q^2)}{e_f^2 x}$ , where  $F_2^{f\text{T(L)}}(x, Q^2)$  is the conventional structure function for a target with definite helicity T,L and includes the contribution of the unresolved photon.  $e_f$  is the electric charge of flavour  $f$ . Within the parton model these structure functions describe the flavour singlet quark distributions  $S_\gamma^{\text{T(L)}}(x, Q^2, P^2) \equiv \sum_{f=1}^{N_f} (q_f^{\text{T(L)}} + \bar{q}_f^{\text{T(L)}})$ . The distribution of gluons is then  $G_\gamma^{\text{T(L)}}(x, Q^2, P^2) = \sum_f G_\gamma^{f\text{T(L)}}(x, Q^2, P^2)$ , where we account for the fact that the gluon distribution may differ for different flavours.

Let us generalize the MSR for a virtual photon target in QED with one lepton flavour. In the lowest order of perturbation theory in  $\alpha_{\text{em}}$ , the photon structure functions are given by the imaginary part of the sum of the box diagrams of Fig.1, for both the real and the virtual target photons. We work in the Bjorken limit  $Q^2 \gg \mu^2, P^2$ ;  $x$  is fixed, with  $\mu$  the mass of the lepton. We start our considerations from the case of a longitudinally polarized photon target. The polarization vector of the longitudinal polarization is  $e_L = (p_3, \mathbf{0}, p_0)/\sqrt{p_3^2 - p_0^2}$ , where  $(p_0, 0, 0, p_3)$  is four momentum of target photon. We denote by axis 3 as the direction of the momentum of the target photon. Using conservation of the electromagnetic current  $p_\mu J_\mu^{\text{em}} = 0$  we obtain,

$$e_\mu^L e_\nu^L M_{\mu\nu}^{\alpha\beta} = P^2 \frac{M_{00}^{\alpha\beta}}{p_3^2}, \quad (1)$$

where  $M_{\mu\nu}^{\alpha\beta}$  is the amplitude of  $\gamma^*(q)\gamma^*(p)$  scattering. The upper indices correspond to the polarization of the projectile photon  $\gamma^*(q)$  and bottom indices correspond to polarization of target photon  $\gamma^*(p)$ . The sum of Feynman diagrams for  $M_{00}^{\alpha\beta}$  is superconvergent. It is thus legitimate to apply the traditional machinery of the parton model, the Wilson Operator Product Expansion (OPE), to  $M_{00}^{\alpha\beta}/p_3^2$  and to use the conservation of the energy-momentum tensor to deduce the MSR. We do not decompose the scattering amplitude into independent Lorentz invariant structures, since it contains more independent invariant functions than for a nucleon target, as we do not sum over the polarization of the target photon. We derive the sum rule for the structure function of the photon target,

$$\int_0^1 x dx \frac{\nu}{p_3^2} \left( \frac{P^2 M_{00}^{33}}{p_3^2} \right) = \int_0^1 dx x S^L(x, Q^2, P^2) = -\frac{d\pi(P^2)}{d \ln P^2}. \quad (2)$$

Here  $\pi(P^2)$  is directly expressed through the polarization operator  $\Pi_{\mu\nu}$  of a target photon  $\gamma^*(P^2)$ ,

$$\Pi_{\mu\nu} = (p^2\delta_{\mu\nu} - p_\mu p_\nu)\pi(p^2). \quad (3)$$

The relationship between the amplitude  $M_{\mu\nu}^{\alpha\beta}$  and the structure function follows from the calculation of Feynman diagrams in the reference frame where the momentum of the target  $p \rightarrow \infty$  but  $q_0$  is small [1]. We choose to work in the center of mass system of the projectile electron and the target photon  $\gamma^*(P^2)$ . In this frame  $q = ((\nu + Q^2)/4|\mathbf{p}|, \mathbf{q}_t, (\nu - Q^2)/4|\mathbf{p}|)$ . Thus the MSR for the electron-positron distributions within the longitudinally polarized photon target is,

$$\zeta_L^{-1} \equiv \int_0^1 dx x S_e^L(x, Q^2, P^2) = -\frac{d\pi(P^2)}{d\ln(P^2)}. \quad (4)$$

Let's now turn to the case of a transversely polarized virtual photon. A naïve application of the impulse approximation leads to the conventional MSR where in difference from Eq.(4) the right hand side is given by the normalization of the photon wave function, *i.e.* by the probability of the target photon to be in an  $e^+e^-$  configuration  $Z_{em}^{-1}$ . The renormalization “constant”  $Z_T^{-1} \equiv Z_{em}^{-1}(P^2) = \frac{d}{dP^2}[P^2\pi(P^2)]$  of the virtual photon  $\gamma^*(P^2)$  is logarithmically ultraviolet divergent. Thus the parton model is applicable for the calculation of the difference between PD in target photons with different virtualities (which is ultraviolet finite) but not for PD themselves,

$$\begin{aligned} \zeta_T^{-1} &\equiv \int_0^1 dx x [S_e^T(x, Q^2, 0) - S_e^T(x, Q^2, P^2)] \\ &= Z_{T,em}^{-1}(0) - Z_{T,em}^{-1}(P^2) = \frac{d}{dP^2}\{P^2[\pi(0) - \pi(P^2)]\}. \end{aligned} \quad (5)$$

For the sake of generalization to QCD it is instructive to explain the problem in terms of the parton model description. The parton wave function of the photon  $\Psi_{\gamma \rightarrow e\bar{e}}(x_1, x_2, k_t)$  [2] is given by the electromagnetic transition  $\gamma^* \rightarrow e\bar{e}$  which includes the energy denominator. Here  $x_i(k_t)$  is the light-cone fraction of the photon momentum (transverse momentum) carried by the electron and the positron. For large  $k_t^2$  the wave function is  $|\Psi|^2 \sim 1/k_t^2$  and therefore  $\int |\Psi|^2 d^2k_t$  diverges logarithmically. If  $k_t^2 \gtrsim Q^2$ , the probing photon interacts coherently with the  $e\bar{e}$  pair in a target photon. There is a destructive interference between the diagrams for a structure function where the virtual photon interacts with one parton (parton model contribution) and the diagrams where the virtual photon interacts with both constituents of the photon. The cancellation between these diagrams, the charge screening phenomenon, effectively cuts the integration over  $k_t^2$  at a value  $\sim Q^2$ . So a correct treatment of the contribution of the unresolved photon leads to a finite value of the matrix element, but the impulse approximation and therefore the momentum sum rule are lost. At the same time the unresolved photon contribution in Eq.(5) is cancelled since it is a high transverse momentum contribution and it does not depend on virtuality of the target photon (within the power accuracy over  $\frac{P^2}{Q^2}$ ).

It is easy to demonstrate that the above reasoning agrees with results of the most complete calculation of box diagrams with virtual photons [3].

To generalize above results to QCD let's consider now the case of an unpolarized target photon. The standard definition of the photon structure function  $F_2 = \frac{1}{2} \sum_\lambda M_{\mu\nu}^{33} e_\lambda^\mu e_\lambda^\nu \nu / p_3^2$

(see for example the first paper of Ref. [4]) leads to the definition of PD for unpolarized virtual photon as follows:

$$S_{\gamma^*(P^2)}(Q^2, P^2, x) = S^T \gamma^*(P^2)(Q^2, P^2, x) - \frac{1}{2} S^L \gamma^*(P^2)(Q^2, P^2, x). \quad (6)$$

And then the MSR for the case of parton distributions in the unpolarized virtual photon has the form,

$$\int_0^1 x [S_{\gamma^*(P^2)}(x, Q^2, P^2) + G_{\gamma^*(P^2)}(x, Q^2, P^2)] dx = \zeta_T(Q^2, P^2) - \frac{1}{2} \zeta_L(Q^2, P^2), \quad (7)$$

where  $\zeta_T$  and  $\zeta_L$  are given in QED by Eqs.(4,5). Let us calculate  $\zeta_T$  and  $\zeta_L$  in QCD. The important difference between QED and QCD is that in QCD the constituents are quarks and gluons and that the box diagrams do not account for the full structure functions of a (quasi)real photon.

The derivation of the MSR in QCD consists of two steps. The first step is to apply OPE, the QCD improved parton model, to the difference of the structure functions of target photons with virtualities  $P^2$  and  $K^2$ . In this difference the contribution of the unresolved photon into MSR is canceled out as in QED. This cancellation is evident in terms of Feynman diagrams since the contribution of the unresolved photon corresponds to virtualities of partons  $\sim Q^2 \gg P^2, K^2$ . Therefore the contribution of unresolved photon is independent on the virtuality of target photon. As a result the validity of the QCD improved parton model approximation in calculating the MSR for the difference of structure functions of photon targets with different virtualities can be justified. For the leading twist contribution the MSR for the difference of structure functions has the same form as in QED (see Eqs.(4,5)),

$$\zeta_L^{-1}(Q^2, P^2) = -\frac{d\pi_{\text{had}}(P^2)}{d \ln P^2} \quad (8)$$

$$\zeta_T^{-1}(Q^2, P^2) - \zeta_T^{-1}(Q^2, K^2) = \frac{d}{dP^2} [P^2 \pi_{\text{had}}(P^2)] - \frac{d}{dK^2} [K^2 \pi_{\text{had}}(K^2)]. \quad (9)$$

Here  $\zeta_{T(L)}^{-1}$  is the normalization of parton distributions for transversely (longitudinally) polarized photons and includes now the quark, antiquark and gluon contributions,

$$\begin{aligned} \zeta_{L(T)}^{-1}(Q^2, P^2) = \\ \int_0^1 dx x [S_{\gamma}^{L(T)}(x, Q^2, P^2) + G_{\gamma}^{L(T)}(x, Q^2, P^2)]. \end{aligned} \quad (10)$$

and the  $\pi(P^2) \equiv \pi_{\text{had}}(P^2)$  is the hadronic contribution to the renormalized photon vacuum polarization operator. It follows from the above discussion that the precision of Eqs.(8, 9) is the same as that for the factorization theorem in QCD.

One of the practical applications of Eqs.(8-10) is the possibility to measure the dependence of the fraction of the photon momentum carried by gluons on the virtuality of the photon. This would be feasible if the momentum carried by quarks of the virtual photon could be measured experimentally.

Thus we deduce the momentum sum rule for the photon structure function as follows:

$$\zeta_T^{-1}(Q^2, P^2) = \zeta_T^{-1}(Q^2, K^2) + \frac{d}{dP^2}[P^2\pi_{\text{had}}(P^2)] - \frac{d}{dK^2}[K^2\pi_{\text{had}}(K^2)]. \quad (11)$$

where

$$\pi_{\text{had}}(0) - \pi_{\text{had}}(K^2) = (1/4\pi^2\alpha_{\text{em}})K^2 \int_0^\infty ds \sigma_h(s)/(s + K^2), \quad (12)$$

and  $\sigma_h(s) \equiv \sigma_{e^+e^- \rightarrow \text{hadrons}}(s)$  with  $s$  the c.m.s. energy square.

The second step in the derivation of the MSR is to choose large  $K^2$  such as  $\Lambda_{\text{QCD}}^2 \ll K^2 \ll Q^2$  where it is legitimate (see Ref. [4]) to apply perturbative QCD and asymptotic freedom to calculate  $\zeta_\gamma^{-1}(Q^2, K^2)$ . It follows from the renormalizability of QCD that  $\pi(K^2)$  can be represented at large  $K^2$  as an asymptotic series in powers of  $\alpha_s(K^2)$ ,

$$\pi(0) - \pi(K^2) = \sum_{r=0} [\alpha_s(K^2)]^r [c_r \ln \frac{K^2}{\Lambda_{\text{QCD}}^2} + d_r]. \quad (13)$$

Here  $c_r$  and  $d_r$  are some numerical coefficients. It follows from Eq.(11) that the same decomposition is valid for  $\zeta_T^{-1}(Q^2, K^2)$  since the l.h.s. of Eq.(11) is independent of  $K^2$ . Thus to calculate the r.h.s. of Eq.(11) at large  $Q^2$  it is sufficient to keep in the polarization operator  $\pi(K^2)$  and in  $\zeta_T^{-1}(Q^2, K^2)$  only terms of zero order in  $\alpha_s(K^2)$ . Other terms cancel out in the r.h.s. of Eq.(11). But the lowest order term in  $\alpha_s(K^2)$  for  $\zeta_T^{-1}(Q^2, K^2)$  is given by the sum of QED box diagrams multiplied by the factor  $N_c \sum_f e_f^2$ ,

$$\zeta_T^{-1}(Q^2, K^2) = N_c \sum_f e_f^2 \frac{\alpha_{\text{em}}}{3\pi} \left[ \ln \frac{Q^2}{K^2} - \frac{1}{12} \right]. \quad (14)$$

It is easy to check that the term  $\ln(1/K^2)$  in Eq.(14) is cancelled on the r.h.s. of Eq.(11) with the corresponding term in  $\pi(K^2)$ .

For the practical purposes, Eqs.(11–14) can be simplified since for the production of each flavour  $f$  one can find such a value  $s_0(f)$  that for  $s > s_0$  the contribution of flavour  $f$  in  $\sigma_h(s)$  is given by quark loops without hard QCD radiative corrections. For  $Q^2 \gg s_0(F)$  Eq.(11) has the form,

$$\begin{aligned} \zeta_T^{-1}(Q^2, P^2) = N_c \sum_{f=1}^F e_f^2 \frac{\alpha_{\text{em}}}{3\pi} \left[ \ln \frac{Q^2}{s_0(F) + P^2} - \frac{1}{12} \right. \\ \left. + \frac{s_0(F)}{s_0(F) + P^2} \right] + (1/4\pi^2\alpha_{\text{em}}) \int_0^{s_0(F)} ds s^2 \sigma_h(s)/(s + P^2)^2. \end{aligned} \quad (15)$$

In the above derivation we ignored the threshold effects related to heavy flavour production. So our formulae are applicable for the production of flavours with masses  $M_q^2 \ll Q^2$ .

Our final result is given by the Eq.(7), where  $\zeta_L^{-1}$  is given by Eq.(8) and  $\zeta_T^{-1}$  is given by Eq.(15). In the approximation of flavor  $SU(3)$  symmetry for the parton distributions in a photon we can deduce the MSR for the nonsinglet structure function which is expected to be valid for  $P^2 \ll 4M_c^2$ ,

$$\int_0^1 x [u(x, Q^2, P^2) - d(x, Q^2, P^2)] dx = [\zeta_T^{-1}(Q^2, P^2) - \frac{1}{2} \zeta_L^{-1}(Q^2, P^2) - \zeta_{T,c}^{-1}(Q^2, P^2)] \frac{(e_u^2 - e_d^2)}{\sum_{u,d,s} e_f^2}. \quad (16)$$

By definition  $\zeta_{T,c}^{-1}(Q^2, P^2)$  is the contribution of the charm quark into the normalization of the structure function calculated through box diagrams. We use the fact that only the term  $\sim \ln(Q^2/M_c^2)$  is important and that this term is dominated by the box diagrams.

It is of interest to compare the sum rule in Eq.(15) with some available parametrizations of parton distributions in a real photon. The results are shown in Fig.2. To estimate  $\zeta_T^{-1}(Q^2, 0)$  we use the parametrization of  $\sigma_h(s)$  from Ref. [5]. It accounts for the production of low mass hadron states and describes  $\sigma_h(s)$  at large  $s$  in terms of the parton model contributions with the first order QCD corrections. (Use of the available experimental parametrization of  $\pi_{\text{had}}(p^2)$  from [6] leads to similar results). For the self-consistency of Eq.(15) we neglect the hard QCD corrections to the expression for  $\sigma_h(s)$  in the parametrization of [5]. In the calculations of structure functions and photon vacuum polarization function we also neglect heavy quark (b,t) effects .

It follows from Fig.2 that the QCD sum rule, Eq.(15), predicts smaller second moment of the sum of parton structure functions of a photon as compared to the existing parametrizations [7–10]. To visualize the difference it is convenient to represent the sum rule for  $Q^2 < 10\text{GeV}^2$  in the form,

$$\begin{aligned} \frac{1}{\alpha_{em}} \int_0^1 x [S_{\gamma^*(P^2=0)} + G_{\gamma^*(P^2=0)}] dx \\ = N_c \sum_f e_f^2 \frac{1}{3\pi} \ln \frac{Q^2}{4\text{GeV}^2} + c. \end{aligned}$$

The parametrizations of Ref. [7], of Ref. [9] and of Refs. [8,10] correspond to  $c \approx 4, 2, 1.5$  respectively, while the QCD sum rule deduced in this Letter corresponds to  $c \approx 1$ .

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- <sup>a</sup> E-mail address: frankfurt@tauphy.tau.ac.il
- <sup>b</sup> E-mail address: gurvich@taunivm.tau.ac.il
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# FIGURES

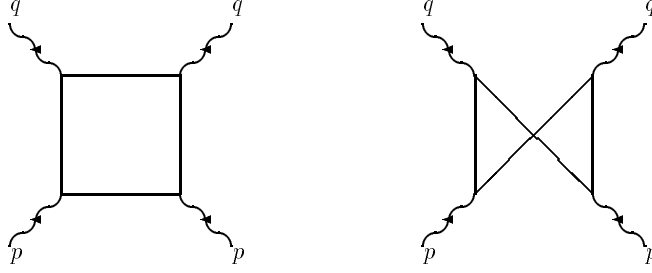


FIG. 1. QED box diagram

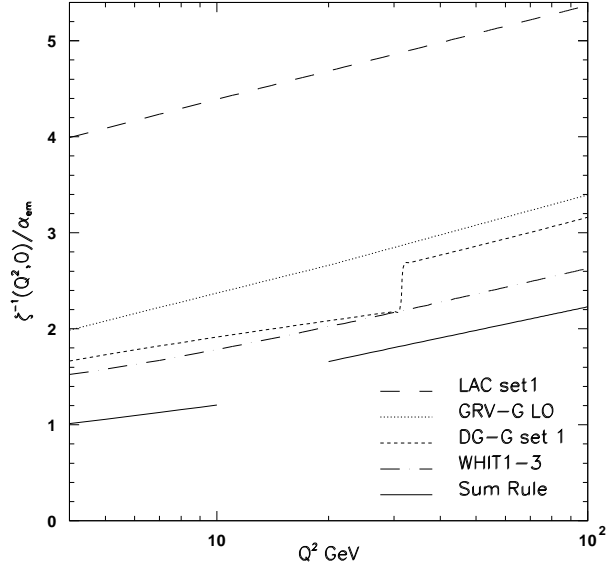


FIG.2. Comparison of the momentum sum rule for the real photon target with parametrizations of the photon PD, LAC from [7], GRV-G LO from [9], DG-G, set 1 from [8] and WHIT1-3 from [10]. The full line is the MSR prediction for  $N_f = 3$ ,  $Q^2 \leq 10\text{GeV}^2$ , and for  $N_f = 4$ ,  $Q^2 \geq 20\text{GeV}^2$ .